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STATIC ANALYSES OF ELASTIC PLATES WITH VOIDS

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Abstract—A general analytical method for plates with arbitrarily-positioned voids is proposed by means of Hamilton's principle. The discontinuous variation of rigidity of the plates due to the voids is expressed continuously by the use of an extended Dirac function, which is defined as a Dirac function existing continuously in a prescribed region. The governing equation for a plate with voids which is composed of an isotropic material is formulated without modifying the rigidity of the plates, as done in the equivalent plate analogy. Static solutions for simply-supported and clamped plates with voids are obtained from the governing equation by means of the Galerkin method. The numerical results obtained from the proposed solutions show good agreement with results obtained from the previous equivalent plate analogy and with results obtained from the finite element method. Also, the exactness of the theory proposed here is established by experiments using acrylic plates.

NOTATION

b_{xij}, b_{yij}	widths in the x and y directions of the i, jth void, respectively
$D(x-x_i)$, $D(y-y_j)$	extended Dirac functions
D_0	flexural rigidity for solid plates without voids
$d(x, y)$	stiffness ratio of plates with voids to plates without voids
E	Young's modulus
$F_{xu}(m, \bar{m}; i)$, $F_{xc}(m, \bar{m}; i)$,	notation with respect to the integral including x
$F_{yu}(n, \bar{n}; j)$, $F_{yc}(n, \bar{n}; j)$	notation with respect to the integral including y
$F_{mn}(k_1, k_2)$, $F_{mn}(k_1, k_2; i)$	notation with respect to the integral
f_{mn}, f_{xm}, f_{yn}	shape functions
G	shear modulus of an isotropic material
h_0	plate thickness
h_{ij}	height of the i, jth void
l_x, l_y	span lengths in the x and y directions
M_x, M_y	bending moments
M_{xy}	twisting moment
p, p_0	external lateral loads and uniform load, respectively
Q_x, Q_y	transverse shear forces
T	kinetic energy
U	strain energy
V	potential energy produced by external loads
V_x, V_y	vertical edge forces
w	lateral deflection on the middle surface
α	ratio of span lengths l_y to l_x
δ	variational operator
$\delta(x-x_i)$, $\delta(y-y_j)$	Dirac functions
κ_x, κ_y	curvature changes of middle surface
κ_{xy}	twisting of the plate
ν	Poisson's ratio
ξ, η	supplementary variables of x and y, respectively
ρ	mass density of plates with voids
σ_x, σ_y	normal stress components
τ	shear stress.

1. INTRODUCTION

For reasons of lightness and structural efficiency and in order to guarantee enough space for equipment, plates with voids are often used. These are called multi-cell slabs with transverse diaphragms, or voided slabs, or cellular slabs, depending upon the shape and size of the voids used. Most methods for the analysis of plates with voids are based on the equivalent plate analogy. With this analogy, even if a plate with voids is composed of an isotropic material, the equivalent plate becomes an orthotropic plate, because the bending rigidity and torsional rigidity are different in different directions owing to the existence of the voids. A number of authors have proposed rigidity coefficients to enable the determination of overall effects. Crisfield and Twemlow (1971) proposed a full equivalent anisotropic plate solution for cellular structures by means of the finite element method, in which the transverse Poisson effect is included. Elliott and Clark (1982) analyzed a slab with one-way circular voids. Cope *et al.* (1973) analyzed a cellular bridge deck by means of a two-dimensional finite element solution for the equivalent shear-weak slab. For cellular slabs, Holmberg (1960), Sawko and Cope (1969) and Elliott (1978) proposed rigidity coefficients. Szilard (1974) and Cope and Clark (1984) summarized previous results for various plates with voids. However, the above-mentioned equivalent approaches have the following faults.

(1) Since the rigidity of plates with voids is determined independently of the position of the voids, application of the theory is restricted only to plates with many voids of the same cross-section, spaced uniformly. Hence it cannot apply to plates with irregularly-spaced voids and/or with voids of different cross-sections.
(2) Local variations of stress couples due to the existence of voids cannot be expressed.

On the other hand, although analyses based on the finite element method for plates with voids are effective, much numerical calculation is needed. A general and simple analytical method usable in both the preliminary and final stages of the design of a plate with voids is desired. However, as mentioned above, a general analytical method for plates with arbitrarily-positioned voids has not been established.

The purpose of this paper is to propose a general method for plates with arbitrarily-positioned voids. The discontinuous variation of rigidity of such plates due to the voids is expressed as a continuous function by means of an extended Dirac function. The extended Dirac function is defined as a Dirac function existing continuously in a prescribed region. For the current problem, the extended Dirac function has a value in the region where voids exist, and replaces the discontinuous variation in the rigidity of the plates due to the voids with a continuous function; it is therefore effective in presenting a general analytical method for plates with arbitrarily-positioned voids. The theory of plates with voids is formulated without modifying the rigidity of the plates, as done in the equivalent plate analogy. The author (Takabatake, 1987, 1988) has demonstrated the effectiveness of the extended Dirac function for bending and torsional analyses of tube systems and for lateral buckling of I beams with web stiffeners and batten plates.

In this paper, the general governing equations for rectangular plates with voids are proposed by using Hamilton's principle. Then static solutions for simply-supported and clamped plates are presented by means of the Galerkin method. Finally, the exactness of the proposed solutions is established from numerical and experimental results.

2. GOVERNING EQUATIONS OF PLATES WITH VOIDS

Consider a rectangular plate with arbitrarily-positioned voids, as shown in Fig. 1. A Cartesian coordinate system x, y, z is employed. Assume that each void is a rectangular parallelepiped whose ridgelines are parallel to the x- or y-axis and which is symmetrically positioned with respect to the middle plane of the plate, as shown in Fig. 2. The position of the i,jth void is indicated by the coordinate value (x_i, y_j) of the midpoint of the void, the widths in the x and y directions of the void are $b_{xi,j}$ and $b_{yi,j}$, respectively, and its height is $h_{i,j}$. The size and position of each void are arbitrary except for the assumptions mentioned above.

Consider the bending of isotropic plates to small deformations, and assume the validity of the Kirchhoff–Love plate theory for the current problem. Hence the transverse shear deformation is neglected. The assumption used here may be effective for structures like floors, roofs, bridges, etc., because the height of the voids is relatively small.

Thus, the shape of a plate with voids is adequately defined by describing the geometry of its middle surface, which is a surface that bisects the plate thickness h_0 at each point. The

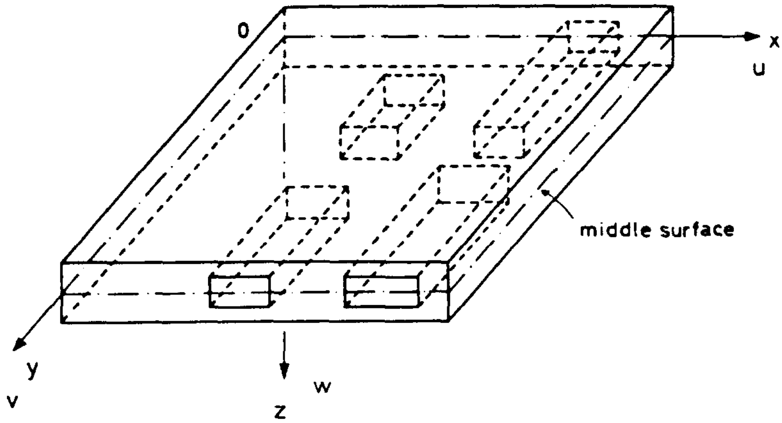


Fig. 1. Coordinates of a rectangular plate with voids.

governing equation of plates with voids is proposed by means of the following Hamilton's principle:

$$\delta I = \delta \int_{t_2}^{t_1} (T - U - V)\,\mathrm{d}t = 0 \tag{1}$$

in which T is the kinetic energy, U is the strain energy, V is the potential energy produced by the external loads, and δ is the variational operator taken during the indicated time interval.

The strain energy U for the current problem is given by

$$U = \frac{1}{2}\int\int [M_x\kappa_x + M_y\kappa_y + 2M_{xy}\kappa_{xy}]\,\mathrm{d}x\,\mathrm{d}y \tag{2}$$

in which κ_x, κ_y and κ_{xy} are the curvatures and twist of the deflected middle surface, with M_x, M_y and M_{xy} the bending and twisting moments per unit width, respectively given by

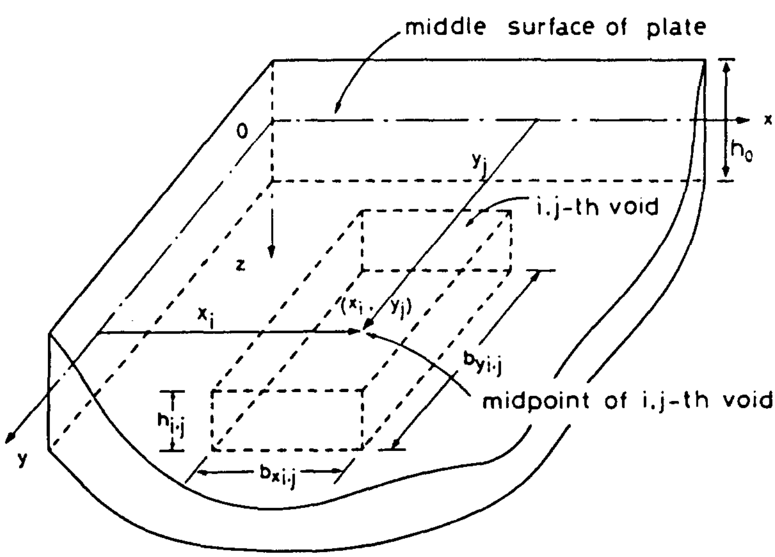


Fig. 2. Details of a void.

$$\left.\begin{aligned} M_x &= \int \sigma_x z \, \mathrm{d}z \\ M_y &= \int \sigma_y z \, \mathrm{d}z \\ M_{xy} &= M_{yx} = \int \tau z \, \mathrm{d}z \end{aligned}\right\} \tag{3}$$

in which σ_x and σ_y are the normal stress components, $\tau(=\sigma_{xy} = \sigma_{yx})$ is the shear stress, and z is measured from the middle surface of the plate. From Szilard (1974), the stresses σ_x, σ_y and τ for isotropic plates can be expressed in terms of the lateral deflections w on the middle surface of the plates:

$$\left.\begin{aligned} \sigma_x &= -\frac{Ez}{1-\nu^2}(w_{,xx} + \nu w_{,yy}) \\ \sigma_y &= -\frac{Ez}{1-\nu^2}(w_{,yy} + \nu w_{,xx}) \\ \tau &= -2Gzw_{,xy} \end{aligned}\right\} \tag{4}$$

in which E is the Young's modulus of the isotropic material, G is the shear modulus of the isotropic material, and ν is Poisson's ratio. The suffixes x and y after the commas indicate partial differentiation with respect to x and y, respectively. From eqns (3) and (4) the bending moment M_x may be written as

$$M_x = -\frac{E}{1-\nu^2}(w_{,xx} + \nu w_{,yy}) \int z^2 \, \mathrm{d}z. \tag{5}$$

At a section where a void exists, calculation of the above integral must be amended to exclude the void, i.e.

$$\int z^2 \, \mathrm{d}z = \int_{-h_0/2}^{h_0/2} z^2 \, \mathrm{d}z - \int_{-h_1/2}^{h_1/2} z^2 \, \mathrm{d}z \tag{6}$$

in which h_0 is the thickness of solid plates and h_1 is the height of the void. h_1 is a function of x and y. At all points in the region where the i,jth void exists, the relation $h_1 = h_{i,j}$ is valid. Hence, $h_1(x, y)$ can generally be expressed by

$$h_1(x, y) = \sum_{i=1}^{m^*} \sum_{j=1}^{n^*} h_{i,j} D(x - x_i) D(y - y_j) \tag{7}$$

in which Σ is the sum for the total number of voids in the plates, m^* and n^* indicate the final numbers of voids in position counting from $i = 1$ and $j = 1$, respectively, and $D(x - x_i)$ and $D(y - y_j)$ are extended Dirac functions. The extended Dirac function $D(x - x_i)$ is defined as a function where the Dirac function $\delta(x - \xi)$ exists continuously in the x direction through the i,jth void, namely the region from $x_i - b_{xi,j}/2$ to $x_i + b_{xi,j}/2$, in which ξ can take values continuously from $x_i - b_{xi,j}/2$ to $x_i + b_{xi,j}/2$. Similarly, $D(y - y_j)$ is a function where the Dirac function $\delta(y - \eta)$ exists continuously in the y direction through the i,jth void, namely the region from $y_j - b_{yi,j}/2$ to $y_j + b_{yi,j}/2$, in which η can take values continuously from $y_j - b_{yi,j}/2$ to $y_j + b_{yi,j}/2$. Briefly, the extended Dirac function D is considered as the sum of the Dirac function δ distributed continuously in a prescribed region. Hence

$$D(x-x_i) = \begin{cases} 1 & \text{for} \quad x_i - \dfrac{b_{xi,j}}{2} < x < x_i + \dfrac{b_{xi,j}}{2} \\ 0 & \text{for all others} \end{cases}$$

$$D(y-y_j) = \begin{cases} 1 & \text{for} \quad y_j - \dfrac{b_{yi,j}}{2} < y < y_j + \dfrac{b_{yi,j}}{2} \\ 0 & \text{for all others.} \end{cases} \tag{8}$$

Takabatake (1987, 1988) has demonstrated the effectiveness of the extended Dirac function for beam problems. The details and employment of the Dirac function are given in Mikusinski and Sikorski (1957) and Frýba (1972), respectively.

Now, substitution of eqn (7) into eqn (6) gives

$$\int z^2 \, dz = \frac{1}{12}\left[h_0^3 - \sum_{i=1}^{m^*} \sum_{j=1}^{n^*} (h_{i,j})^3 D(x-x_i) D(y-y_j)\right]. \tag{9}$$

Hence eqn (5) may be written as

$$M_x = -D_0 d(x,y)[w_{,xx} + \nu w_{,yy}] \tag{10}$$

in which D_0 is the flexural rigidity of a solid plate neglecting voids and is given by $Eh_0^3/12(1-\nu^2)$, and $d(x, y)$ is given by

$$d(x,y) = 1 - \sum_{i=1}^{m^*} \sum_{j=1}^{n^*} \alpha_{ij} D(x-x_i) D(y-y_j) \tag{11}$$

in which α_{ij} is defined as

$$\alpha_{ij} = \left(\frac{h_{i,j}}{h_0}\right)^3. \tag{12}$$

Similarly, the bending moment M_y and twisting moment M_{xy} can be written as

$$M_y = -D_0 d(x,y)[w_{,yy} + \nu w_{,xx}] \tag{13}$$

$$M_{xy} = -(1-\nu) D_0 d(x,y) w_{,xy}. \tag{14}$$

The curvature changes κ_x, κ_y and κ_{xy} are defined as

$$\left.\begin{aligned} \kappa_x &= -w_{,xx} \\ \kappa_y &= -w_{,yy} \\ \kappa_{xy} &= -w_{,xy} \end{aligned}\right\}. \tag{15}$$

The theory including the transverse shear deformation will be easily derived by employing the curvature–displacement relations of the Mindlin plate theory or of a high-order deformational mode in place of eqn (15). Hence, substituting eqns (10) and (13)–(15) into eqn (2), the strain energy U becomes

$$U = \frac{D_0}{2}\int\int d(x,y)[(w_{,xx})^2+(w_{,yy})^2+2\nu w_{,xx}w_{,yy}+2(1-\nu)(w_{,xy})^2]\,\mathrm{d}x\,\mathrm{d}y. \tag{16}$$

Next, the potential energy V produced by the external lateral loads p becomes

$$V = -\int\int pw\,\mathrm{d}x\,\mathrm{d}y. \tag{17}$$

Neglecting the effect of rotatory inertia, the kinetic energy T is

$$T = \frac{1}{2}\int\int \rho h(x,y)(\dot{w})^2\,\mathrm{d}x\,\mathrm{d}y \tag{18}$$

in which the dot indicates differentiation with respect to time, ρ is the mass density of the plate with voids, and the notation $h(x, y)$ is defined as

$$h(x,y) = h_0\left[1-\sum_{i=1}^{m^*}\sum_{j=1}^{n^*} h_{i,j}D(x-x_i)D(y-y_j)\right]. \tag{19}$$

Substitution of eqns (16)–(18) into eqn (1) yields

$$\delta I = \int_{t_1}^{t_2}\left\{\int\int F_1\,\delta w\,\mathrm{d}x\,\mathrm{d}y+\int F_2\,\delta w\Big|_0^{l_x}\mathrm{d}y-\int F_3\,\delta w_{,x}\Big|_0^{l_x}\mathrm{d}y\right.$$
$$\left.+\int F_4\,\delta w\Big|_0^{l_y}\mathrm{d}x-\int F_5\,\delta w_{,y}\Big|_0^{l_y}\mathrm{d}x-2(1-\nu)F_6\,\delta w\Big|_0^{l_x}\Big|_0^{l_y}\right\}\mathrm{d}t = 0 \tag{20}$$

in which F_1–F_6 are given by the LHS of eqns (21) and $(22)_2$–$(26)_2$. Here l_x and l_y are the span lengths in the x and y directions of the plate, respectively.

From eqn (20), the differential equation of motion can be obtained

$$\frac{\rho h(x,y)\ddot{w}}{D_0}+[dw_{,xx}]_{,xx}+[dw_{,yy}]_{,yy}+\nu[dw_{,yy}]_{,xx}+\nu[dw_{,xx}]_{,yy}+2(1-\nu)[dw_{,xy}]_{,xy}-\frac{p}{D_0}=0 \tag{21}$$

together with the associated boundary conditions

$$w=0 \quad \text{or} \quad D_0[dw_{,xx}]_{,x}+\nu D_0[dw_{,yy}]_{,x}+2(1-\nu)D_0[dw_{,xy}]_{,y}=0 \tag{22}$$

$$w_{,x}=0 \quad \text{or} \quad D_0[dw_{,xx}+\nu dw_{,yy}]=0 \tag{23}$$

at $x = 0$ and l_x; and

$$w=0 \quad \text{or} \quad D_0[dw_{,yy}]_{,y}+\nu D_0[dw_{,xx}]_{,y}+2(1-\nu)D_0[dw_{,xy}]_{,x}=0 \tag{24}$$

$$w_{,y}=0 \quad \text{or} \quad D_0[dw_{,yy}+\nu dw_{,xx}]=0 \tag{25}$$

at $y = 0$ and l_y; and

$$w=0 \quad \text{or} \quad D_0 dw_{,xy}=0 \tag{26}$$

at the corners.

For solid plates without voids, $d(x, y)$ becomes 1 and the governing equations proposed here reduce to the general equations for rectangular solid plates.

3. STATIC ANALYSES TO RECTANGULAR PLATES WITH VOIDS

The governing equations for rectangular plates with voids have been proposed. Now consider the static solutions for simply-supported and clamped plates by means of the Galerkin method. The deflections $w(x, y)$ can be expressed by a power-series expansion as follows:

$$w(x, y) = \sum_{m=1} \sum_{n=1} w_{mn} f_{mn}(x, y) \tag{27}$$

in which the f_{mn} are shape functions satisfying the specified boundary conditions. The following functions represent f_{mn} for simply-supported and clamped plates:

$$f_{mn}(x, y) = \sin\frac{m\pi x}{l_x}\sin\frac{n\pi y}{l_y} \quad \text{for simply supported plates}$$

$$f_{mn}(x, y) = \sin\frac{\pi x}{l_x}\sin\frac{m\pi x}{l_x}\sin\frac{\pi y}{l_y}\sin\frac{n\pi y}{l_y} \quad \text{for clamped plates.} \tag{28}$$

The Galerkin equation for static problems can be written as

$$\int_0^{l_x}\int_0^{l_y} Q\,\delta w\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{29}$$

in which Q is the equation neglecting the inertia term in eqn (21). Substituting eqn (27) into eqn (29), the Galerkin equations become

$$\delta w_{\bar{m}\bar{n}}: \sum_{m=1}\sum_{n=1} w_{mn}\int_0^{l_x}\int_0^{l_y}\{[df_{mn,xx}]_{,xx}+[df_{mn,yy}]_{,yy}+\nu[df_{mn,yy}]_{,xx}$$
$$+\nu[df_{mn,xx}]_{,yy}+2(1-\nu)[df_{mn,xy}]_{,xy}\}f_{\bar{m}\bar{n}}\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_0^{l_x}\int_0^{l_y}\frac{p}{D_0}f_{\bar{m}\bar{n}}\,\mathrm{d}x\,\mathrm{d}y. \tag{30}$$

Then, the integral calculation including the extended Dirac function $D(x-x_i)$ can be written as

$$\int_0^{l_x} D(x-x_i)f(x)\,\mathrm{d}x = \int_{x_i-(h_{x,i}/2)}^{x_i+(h_{x,i}/2)}[\delta(x-\xi)f(x)\,\mathrm{d}x]\,\mathrm{d}\xi = \int_{x_i-(h_{x,i}/2)}^{x_i+(h_{x,i}/2)} f(\xi)\,\mathrm{d}\xi \tag{31}$$

in which ξ is a supplementary variable of x. Similarly,

$$\int_0^{l_y} D(y-y_i)f(y)\,\mathrm{d}y = \int_{y_i-(h_{y,i}/2)}^{y_i+(h_{y,i}/2)} f(\eta)\,\mathrm{d}\eta \tag{32}$$

in which η is a supplementary variable of y. The nth derivatives of the extended Dirac functions can therefore be expressed as

$$\left.\begin{aligned}\int_0^{l_x} D^{(n)}(x-x_i)f(x)\,\mathrm{d}x &= \int_{x_i-(h_{xi,j}/2)}^{x_i+(h_{xi,j}/2)} (-1)^n f^{(n)}(\xi)\,\mathrm{d}\xi \\ \int_0^{l_y} D^{(n)}(y-y_j)f(y)\,\mathrm{d}y &= \int_{y_j-(h_{yi,j}/2)}^{y_j+(h_{yi,j}/2)} (-1)^n f^{(n)}(\eta)\,\mathrm{d}\eta\end{aligned}\right\} \tag{33}$$

in which superscripts enclosed within parentheses indicate the differential order.

When the conditions $h_{xi,j} \ll l_x$ and $h_{yi,j} \ll l_y$ are satisfied, the extended Dirac functions $D(x-x_i)$ and $D(y-y_j)$ are approximately related to the Dirac functions $\delta(x-x_i)$ and $\delta(y-y_j)$ by:

$$\left.\begin{aligned}D(x-x_i) &\fallingdotseq h_{xi,j}\,\delta(x-x_i)\\ D(y-y_j) &\fallingdotseq h_{yi,j}\,\delta(y-y_j)\end{aligned}\right\}. \tag{34}$$

To simplify, assume the lateral loads p are a uniform load p_0. Substituting eqn (28) into eqn (30), the Galerkin equations reduce to a system of linear algebraic equations with respect to the displacement coefficients w_{mn}, viz.

$$\delta w_{\bar{m}\bar{n}}: \sum_{m=1} \sum_{n=1} w_{mn} A_{\bar{m}\bar{n}mn} = B_{\bar{m}\bar{n}}. \tag{35}$$

In the above system of linear algebraic equations the row is given by $\bar{m}$ and $\bar{n}$ and the column by m and n.

For simply-supported plates with voids, $A_{\bar{m}\bar{n}mn}$ and $B_{\bar{m}\bar{n}}$ are given by

$$\begin{aligned}
A_{\bar{m}\bar{n}mn} = {} & \pi^4\left[m^2+\left(\frac{n}{\alpha}\right)^2\right]^2 \delta_{m\bar{m}}\delta_{n\bar{n}} - \sum_{i=1}\sum_{j=1} 4\pi^4\alpha_{ij}\Bigg[F_{xss}(m,\bar{m};i)F_{yss}(n,\bar{n};j)\\
& \times\left[m^2+\left(\frac{n}{\alpha}\right)^2\right]^2 - 2m\left[m^2+\left(\frac{n}{\alpha}\right)^2\right][mF_{xss}(m,\bar{m};i)-\bar{m}F_{xcc}(m,\bar{m};i)]F_{yss}(n,\bar{n};j)\\
& -2\left[\left(\frac{n}{\alpha}\right)^3+m^2\frac{n}{\alpha}\right]\left[\frac{n}{\alpha}F_{yss}(n,\bar{n};j)-\frac{\bar{n}}{\alpha}F_{ycc}(n,\bar{n};j)\right]F_{xss}(m,\bar{m};i)\\
& +\left[m^2+\nu\left(\frac{n}{\alpha}\right)^2\right]\{[m^2+\bar{m}^2]F_{xss}(m,\bar{m};i)-2m\bar{m}F_{xcc}(m,\bar{m};i)\}F_{yss}(n,\bar{n};j)\\
& +\left[\left(\frac{n}{\alpha}\right)^2+\nu m^2\right]\left\{\left[\left(\frac{n}{\alpha}\right)^2+\left(\frac{\bar{n}}{\alpha}\right)^2\right]F_{yss}(n,\bar{n};j)-2\frac{n}{\alpha}\frac{\bar{n}}{\alpha}F_{ycc}(n,\bar{n};j)\right\}F_{xss}(m,\bar{m};i)\\
& +2(1-\nu)m\frac{n}{\alpha}[mF_{xss}(m,\bar{m};i)-\bar{m}F_{xcc}(m,\bar{m};i)]\\
& \times\left[\frac{n}{\alpha}F_{yss}(n,\bar{n};j)+\frac{\bar{n}}{\alpha}F_{ycc}(n,\bar{n};j)\right]\Bigg]
\end{aligned} \tag{36}$$

$$B_{\bar{m}\bar{n}} = \begin{cases} \dfrac{16}{\bar{m}\bar{n}\pi^2}\left[\dfrac{p_0 l_x^4}{D_0}\right] & \text{for odd } \bar{m}, \bar{n} \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

in which $\delta_{m\bar{m}}$ and $\delta_{n\bar{n}}$ are the Kronecker deltas, α is the ratio l_y/l_x of the span lengths, and the notations $F_{xss}(m,\bar{m};i)$ and $F_{xcc}(m,\bar{m};i)$ are defined as

$$\begin{Bmatrix} F_{xss}(m,\bar{m};i) \\ F_{xcc}(m,\bar{m};i) \end{Bmatrix} = \frac{1}{l_x}\int_{x_i-(b_{xi,j}/2)}^{x_i+(b_{xi,j}/2)} \begin{Bmatrix} \sin\left(\dfrac{m\pi x}{l_x}\right)\sin\left(\dfrac{\bar{m}\pi x}{l_x}\right) \\ \cos\left(\dfrac{m\pi x}{l_x}\right)\cos\left(\dfrac{\bar{m}\pi x}{l_x}\right) \end{Bmatrix} \mathrm{d}x$$

$$= \frac{1}{(m-\bar{m})\pi}\cos\frac{(m-\bar{m})\pi x_i}{l_x}\sin\frac{(m-\bar{m})\pi b_{xi,j}}{2l_x}(1-\delta_{m\bar{m}}) + \frac{1}{2}\left(\frac{b_{xi,j}}{l_x}\right)\delta_{m\bar{m}}$$

$$\mp \frac{1}{(m+\bar{m})\pi}\cos\frac{(m+\bar{m})\pi x_i}{l_x}\sin\frac{(m+\bar{m})\pi b_{xi,j}}{2l_x}. \tag{38}$$

The notations $F_{yss}(n,\bar{n};j)$ and $F_{ycc}(n,\bar{n};j)$ are obtained by transforming $m\to n$, $\bar{m}\to\bar{n}$, $x_i\to y_j$, $b_{xi,j}\to b_{yi,j}$ and $l_x\to l_y$ in eqn (38).

On the other hand, the expressions for $A_{\bar{m}\bar{n}mn}$ and $B_{\bar{m}\bar{n}}$ for clamped plates are

$$A_{\bar{m}\bar{n}mn} = \pi^4\left[\alpha F_{m\bar{m}}(4,0)F_{n\bar{n}}(0,0) + \frac{2}{\alpha}F_{m\bar{m}}(2,0)F_{n\bar{n}}(2,0) + \left(\frac{1}{\alpha}\right)^3 F_{m\bar{m}}(0,0)F_{n\bar{n}}(4,0)\right]$$

$$-\sum_{i=1}\sum_{j=1}\pi^4\alpha_{ij}\Bigg[\alpha F_{m\bar{m}}(4,0;i)F_{n\bar{n}}(0,0;j) + \frac{2}{\alpha}F_{m\bar{m}}(2,0;i)F_{n\bar{n}}(2,0;j)$$

$$+\left(\frac{1}{\alpha}\right)^3 F_{m\bar{m}}(0,0;i)F_{n\bar{n}}(4,0;j) - 2\alpha[F_{m\bar{m}}(4,0;i)+F_{m\bar{m}}(3,1;i)]F_{n\bar{n}}(0,0;j)$$

$$-2\left(\frac{1}{\alpha}\right)^3[F_{n\bar{n}}(4,0;j)+F_{n\bar{n}}(3,1;j)]F_{m\bar{m}}(0,0;i)$$

$$+\alpha[F_{m\bar{m}}(4,0;i)+2F_{m\bar{m}}(3,1;i)+F_{m\bar{m}}(2,2;i)]F_{n\bar{n}}(0,0;j)$$

$$+\left(\frac{1}{\alpha}\right)^3[F_{n\bar{n}}(4,0;j)+2F_{n\bar{n}}(3,1;j)+F_{n\bar{n}}(2,2;j)]F_{m\bar{m}}(0,0;i)$$

$$-\frac{2}{\alpha}\{[F_{m\bar{m}}(2,0;i)+F_{m\bar{m}}(1,1;i)]F_{n\bar{n}}(2,0;j)+[F_{n\bar{n}}(2,0;j)$$

$$+F_{n\bar{n}}(1,1;j)]F_{m\bar{m}}(2,0;i)\} + \frac{2(1-\nu)}{\alpha}[F_{m\bar{m}}(2,0;i)+F_{m\bar{m}}(1,1;i)][F_{n\bar{n}}(2,0;j)$$

$$+F_{n\bar{n}}(1,1;j)] + \frac{\nu}{\alpha}[F_{m\bar{m}}(2,0;i)+2F_{m\bar{m}}(1,1;i)+F_{m\bar{m}}(0,2;i)]F_{n\bar{n}}(2,0;j)$$

$$+[F_{n\bar{n}}(2,0;j)+2F_{n\bar{n}}(1,1;j)+F_{n\bar{n}}(0,2;j)]F_{m\bar{m}}(2,0;i)\Bigg] \tag{39}$$

$$B_{\bar{m}\bar{n}} = \tfrac{1}{4}\delta_{\bar{m}1}\delta_{\bar{n}1}\alpha\left[\frac{p_0 l_x^4}{D_0}\right] \tag{40}$$

in which $F_{m\bar{m}}(0,0)$, $F_{m\bar{m}}(2,0)$, $\ldots$, $F_{m\bar{m}}(0,0;i)\ldots$ are expressed in general form by

$$\left.\begin{aligned}
F_{m\bar{m}}(k_1,k_2) &= (l_x)^{k_1+k_2-1}\int_0^{l_x} f_{xm}^{(k_1)} f_{x\bar{m}}^{(k_2)}\,\mathrm{d}x \\
F_{n\bar{n}}(k_1,k_2) &= (l_y)^{k_1+k_2-1}\int_0^{l_y} f_{yn}^{(k_1)} f_{y\bar{n}}^{(k_2)}\,\mathrm{d}y \\
F_{m\bar{m}}(k_1,k_2;i) &= (l_x)^{k_1+k_2-1}\int_0^{l_x} D(x-x_i) f_{xm}^{(k_1)} f_{x\bar{m}}^{(k_2)}\,\mathrm{d}x \\
&= (l_x)^{k_1+k_2-1}\int_{x_i-(b_{x,i})/2}^{x_i+(b_{x,i})/2} f_{xm}^{(k_1)}(\xi) f_{x\bar{m}}^{(k_2)}(\xi)\,\mathrm{d}\xi \\
F_{n\bar{n}}(k_1,k_2;j) &= (l_y)^{k_1+k_2-1}\int_{y_j-(b_{y,j})/2}^{y_j+(b_{y,j})/2} f_{yn}^{(k_1)}(\eta) f_{y\bar{n}}^{(k_2)}(\eta)\,\mathrm{d}\eta
\end{aligned}\right\} \tag{41}$$

in which f_{xm} and f_{yn} are the x and y components of the shape function given in eqn (28), viz.

$$\left.\begin{aligned}
f_{xm} &= \sin\frac{\pi x}{l_x}\sin\frac{m\pi x}{l_x} \\
f_{yn} &= \sin\frac{\pi y}{l_y}\sin\frac{n\pi y}{l_y}
\end{aligned}\right\}. \tag{42}$$

Thus, solving eqn (35) for the unknown displacement coefficients w_{mn} and substituting them into eqn (27), the deflections w are obtained. The integrals involving the extended Dirac functions in eqns (36) and (39) have been rigorously calculated on the basis of eqns (31) and (32). However, if the width of each void is small compared with the corresponding span length, the integral calculation is rapidly simplified by the use of the relations given in eqn (34). For example, eqns (38) and (41) are simplified as follows:

$$\left.\begin{aligned}
F_{xs}(m,\bar{m};i) &\fallingdotseq b_{x,i}\sin\left(\frac{m\pi x_i}{l_x}\right)\sin\left(\frac{\bar{m}\pi x_i}{l_x}\right) \\
F_{xc}(m,\bar{m};i) &\fallingdotseq b_{x,i}\cos\left(\frac{m\pi x_i}{l_x}\right)\cos\left(\frac{\bar{m}\pi x_i}{l_x}\right) \\
F_{m\bar{m}}(k_1,k_2;i) &\fallingdotseq b_{x,i}(l_x)^{k_1+k_2-1} f_{xm}^{(k_1)}(x_i) f_{x\bar{m}}^{(k_2)}(x_i) \\
F_{n\bar{n}}(k_1,k_2;j) &\fallingdotseq b_{y,j}(l_y)^{k_1+k_2-1} f_{yn}^{(k_1)}(y_j) f_{y\bar{n}}^{(k_2)}(y_j)
\end{aligned}\right\}. \tag{43}$$

Although the behavior of plates with voids is affected by all the terms of the square matrix $A_{\bar{m}\bar{n}mn}$, the behavior is now dominated by the diagonal terms in the matrix $A_{\bar{m}\bar{n}mn}$. Hence, taking into consideration only the diagonal terms of $A_{\bar{m}\bar{n}mn}$, eqn (35) becomes of uncoupled form. Thus the approximate solutions of w_{mn} are obtained as

$$w_{mn} = \frac{B_{mn}}{A_{mnmn}}. \tag{44}$$

The bending moments M_x and M_y and twisting moment M_{xy} are given by substituting the deflections w into eqns (10), (13) and (14), respectively. The transverse shear forces Q_x and Q_y and vertical edge forces V_x and V_y per unit length are given by

$$\left.\begin{aligned} Q_x &= M_{x,x} + M_{xy,y} \\ Q_y &= M_{y,y} + M_{xy,x} \end{aligned}\right\} \tag{45}$$

$$\left.\begin{aligned} V_x &= Q_x + M_{xy,y} \\ V_y &= Q_y + M_{yx,x} \end{aligned}\right\}. \tag{46}$$

Here the differential $M_{x,x}$ is calculated as

$$M_{x,x} = -D_0\{d(x,y)_{,x}[w_{,xx} + \nu w_{,yy}] + d(x,y)[w_{,xxx} + \nu w_{,xyy}]\}. \tag{47}$$

From eqn (11), the differential of $d(x,y)$ with respect to x is

$$d(x,y)_{,x} = -\sum_{i=1}\sum_{j=1} \alpha_{ij} D(x-x_i)_{,x} D(y-y_j). \tag{48}$$

From Sinozaki *et al.* (1983), the integration involving the differential of the Dirac function is expressed by

$$\int \delta(x-x_i)_{,x} f(x)\,\mathrm{d}x = -\int \delta(x-x_i) f(x)_{,x}\,\mathrm{d}x. \tag{49}$$

Differentiating the above equation with respect to x yields

$$\delta(x-x_i)_{,x} f(x) = -\delta(x-x_i) f(x)_{,x}. \tag{50}$$

For the extended Dirac function eqn (50) may be extended as

$$D(x-x_i)_{,x} f(x) = -D(x-x_i) f(x)_{,x}. \tag{51}$$

Similarly,

$$D(y-y_j)_{,y} f(y) = -D(y-y_j) f(y)_{,y}. \tag{52}$$

The substitution of eqns (11), (48) and (51) into eqn (47) results in

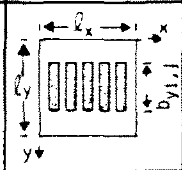
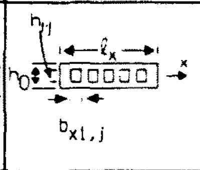
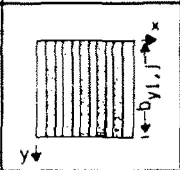
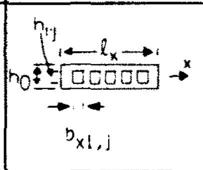
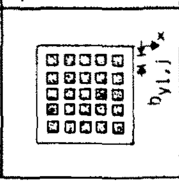
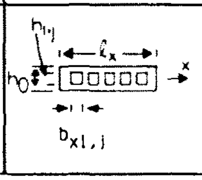
$$M_{x,x} = -D_0[w_{,xxx} + \nu w_{,xyy}]. \tag{53}$$

The result is not affected directly by the extended Dirac functions. Similarly,

$$\left.\begin{aligned} M_{y,y} &= -D_0[w_{,yyy} + \nu w_{,xxy}] \\ M_{xy,x} &= -(1-\nu) D_0 w_{,xxy} \\ M_{xy,y} &= -(1-\nu) D_0 w_{,xyy} \end{aligned}\right\}. \tag{54}$$

Thus the transverse shear forces Q_x and Q_y and the vertical edge forces V_x and V_y become

Table 1. Lists of isotropic rectangular plates with voids

TYPE	PLANE	SECTION	$\frac{h_{i,j}}{h_0}$	$\frac{b_{xi,j}}{\ell_x}$	$\frac{b_{yi,j}}{\ell_y}$	$\alpha=\frac{\ell_y}{\ell_x}$
1			0.5	0.1	0.5	1.0
2			0.5	0.1	1.0	1.0
3			0.5	0.1	0.1	1.0

$$\left.\begin{aligned} Q_x &= -D_0[w_{,xxx}+w_{,xyy}] \\ Q_y &= -D_0[w_{,yyy}+w_{,xxy}] \\ V_x &= -D_0[w_{,xxx}+2w_{,xyy}-\nu w_{,xyy}] \\ V_y &= -D_0[w_{,yyy}+2w_{,xxy}-\nu w_{,xxy}] \end{aligned}\right\}. \tag{55}$$

4. NUMERICAL RESULTS

Static solutions for simply-supported and clamped plates with voids have been presented by means of the Galerkin method. All external terms B_{mn} given in eqns (37) and (40) have the same dimension, $p_0 l_x^4/D_0$. Hence the displacements w, stress couples M_x, M_y and M_{xy} and stress resultants Q_x, Q_y, V_x and V_y can be expressed in nondimensional forms by taking $p_0 l_x^4/D_0$, $p_0 l_x^2$ and $p_0 l_x$ as the units, respectively.

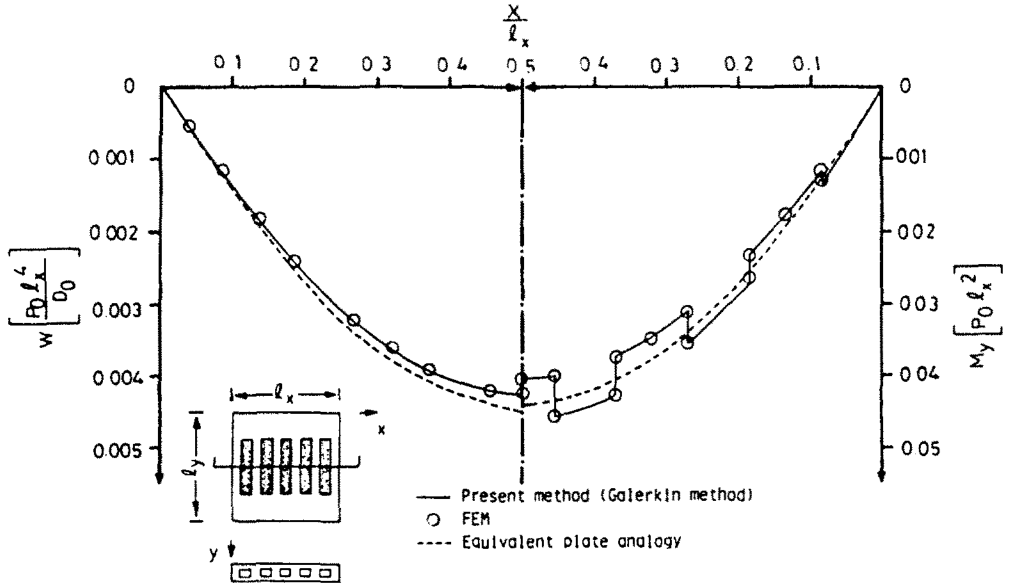


Fig. 3. w and M_y for a simply-supported plate with voids.

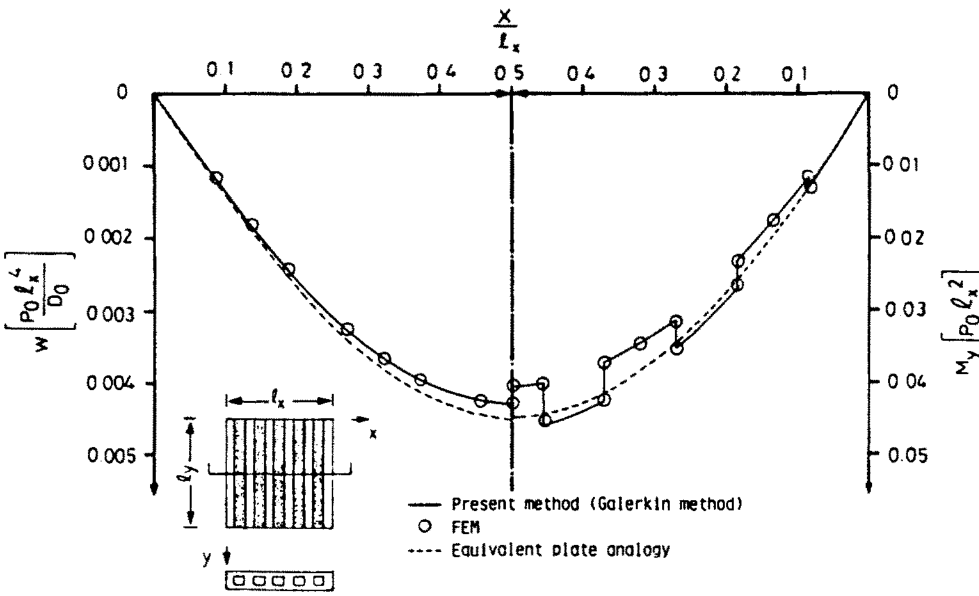


Fig. 4. w and M_y for a simply-supported plate with voids.

Then, in order to examine the proposed solutions, numerical calculations are carried out for three cases as shown in Table 1, in which Poisson's ratio is 0.17. Figures 3–5 and 6–8 show the deflections and bending moments M_y for the three cases of simply-supported and clamped plates with voids, respectively. Numerical results show that, in practice, the differences between the rigorous solutions based on eqn (35) and the approximate solutions based on eqn (44) are negligible. The results obtained from the Galerkin method show good agreement with the results obtained from the finite element method. The finite element method used here is based on isotropic and rectangular plate elements due to Adini–Clough–Melosh, as given by Rao (1982) and Ugural (1981), in which an element with voids includes the effect of the voids, and is independent of FEM-based on equivalent orthotropic plate theory as given by Hinton and Owen (1984). In addition, the numerical results obtained from the equivalent plate analogy by Crisfield and Twemlow (1971) are close to the numerical results of the Galerkin method. However, it is clear that the equivalent plate analogy cannot give good results for all cases and that, especially, the values obtained for

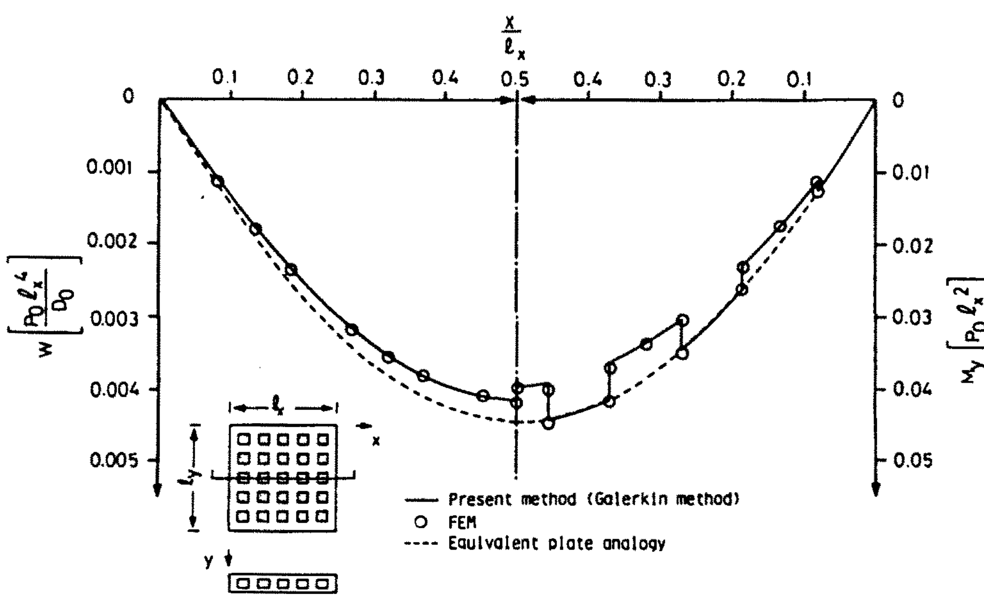


Fig. 5. w and M_y for a simply-supported plate with voids.

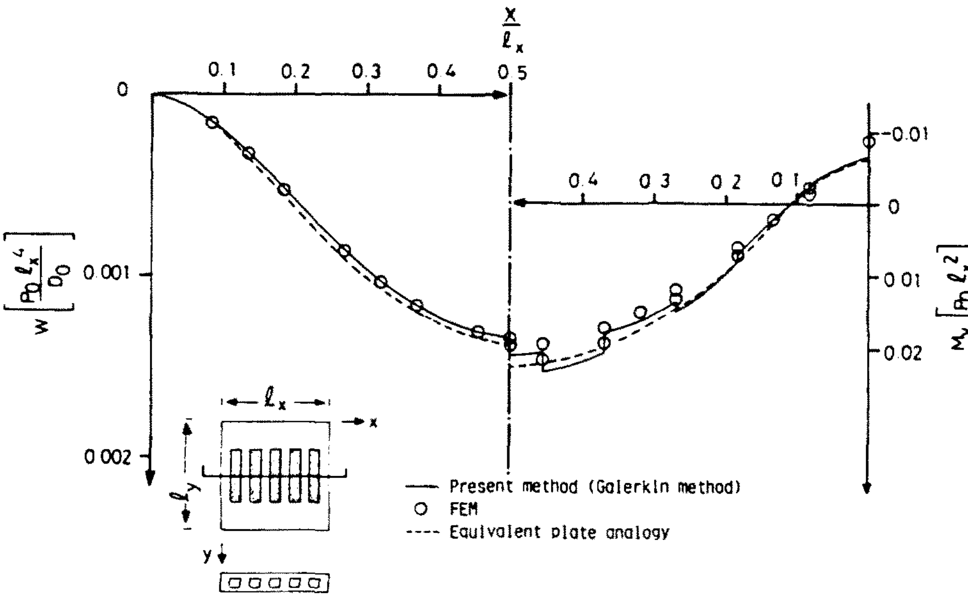


Fig. 6. w and M_y for a clamped plate with voids.

the bending moments M_y indicate mean values including the effect of the local rigidity due to voids. This point must be taken into consideration in designs using the equivalent plate analogy.

5. RELATIONSHIPS BETWEEN THEORETICAL AND EXPERIMENTAL RESULTS

In order to experimentally examine the theory proposed here, static experiments for acrylic plates with voids were carried out for simply-supported and clamped plates. The experimental equipment is shown, in outline, in Fig. 9, in which the span lengths $l_x = l_y = 30$ cm (11.8 in.). Although the positions of the voids in the specimens are the same as the voided plates shown in Table 1, used in the numerical calculations mentioned above, the thickness and the ratios of void size, $h_{i,j}/h_0$, $b_{xi,j}/l_x$ and $b_{yi,j}/l_y$, take the following values:

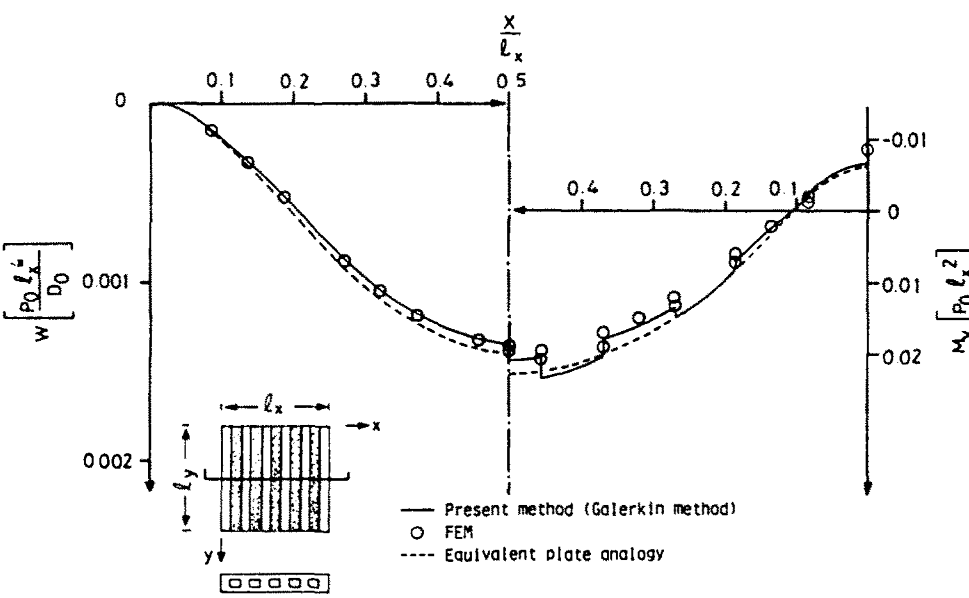


Fig. 7. w and M_y for a clamped plate with voids.

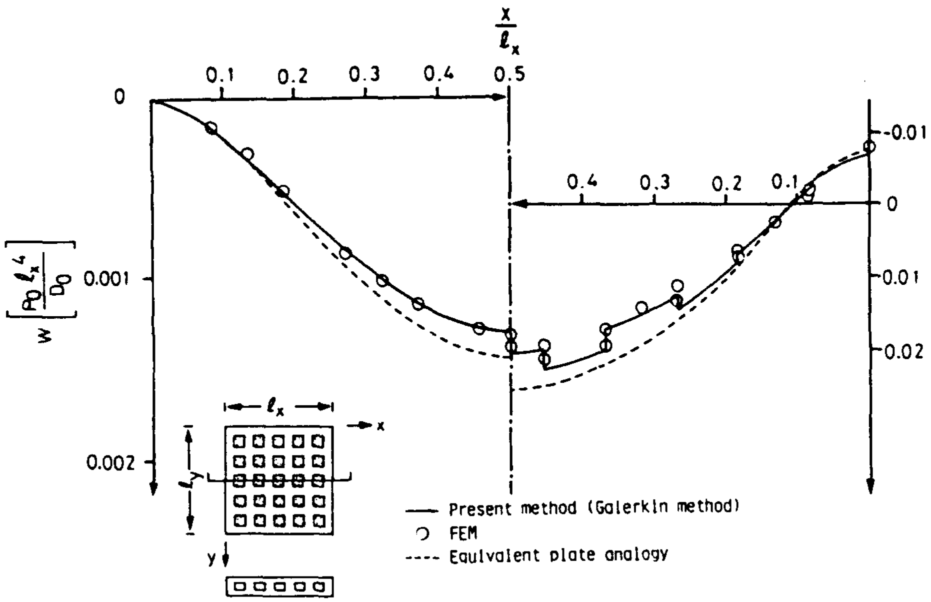


Fig. 8. w and M_y for a clamped plate with voids.

$$\text{Type 0}: h_0 = 0.6\ \text{cm}\ (0.236\ \text{in.})$$

$$\text{Type 1}: h_0 = 0.6\ \text{cm},\quad h_{i,j}/h_0 = 0.33,\quad b_{xi,j}/l_x = 0.1,\quad b_{yi,j}/l_y = 0.5$$
$$\text{Type 2}: h_0 = 0.6\ \text{cm},\quad h_{i,j}/h_0 = 0.33,\quad b_{xi,j}/l_x = 0.1,\quad b_{yi,j}/l_y = 1.0$$
$$\text{Type 3}: h_0 = 0.6\ \text{cm},\quad h_{i,j}/h_0 = 0.33,\quad b_{xi,j}/l_x = 0.1,\quad b_{yi,j}/l_y = 0.1.$$

The Young's modulus and Poisson's ratio of the acrylic plates used are $E = 32{,}700$ kgf cm^{-2} (46.5×10^4 lb in.$^{-2}$) and $\nu = 0.34$, respectively. In order to examine the experimental equipment used, experiments for plates without voids, called Type 0, were carried out, and the experimental results showed good agreement with the theoretical results, as shown in Fig. 10. The relationships between the deflections at the midpoint of the specimens and the lateral uniform load per unit area are shown in Figs 11–13. It follows from these figures that the theory proposed here shows strong agreement in the linear region. Thus, it is shown that the theory proposed here can be applied practically to plates with voids.

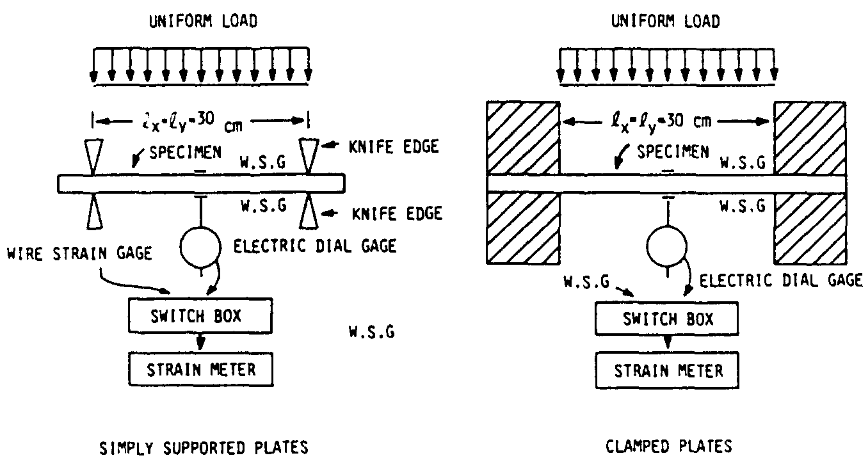


Fig. 9. Outline of the experimental equipment.

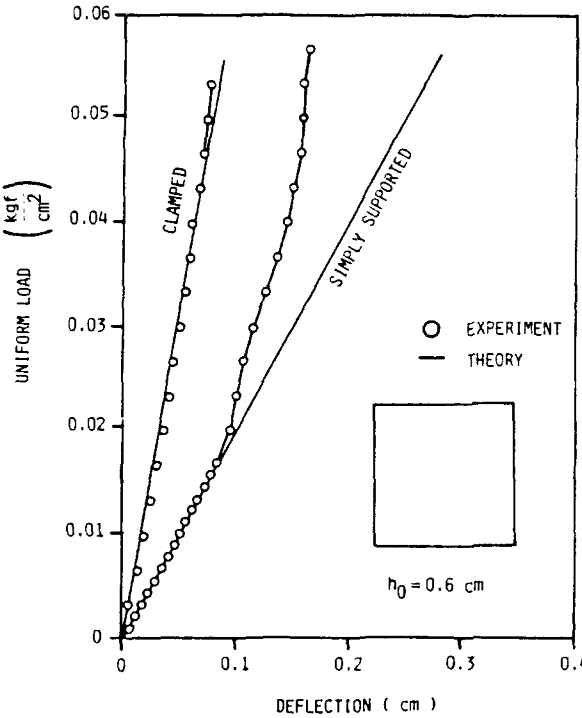


Fig. 10. Relationship between the lateral load and deflection for Type 0 (normal plate).

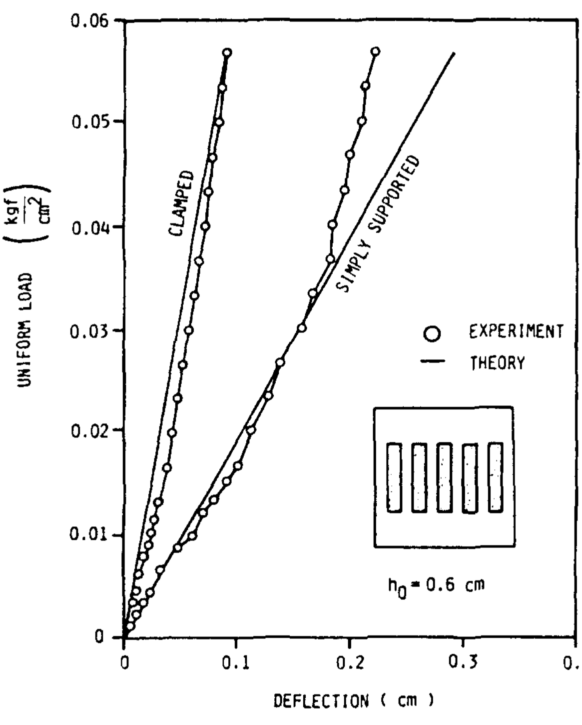


Fig. 11. Relationship between the lateral load and deflection for Type 1.

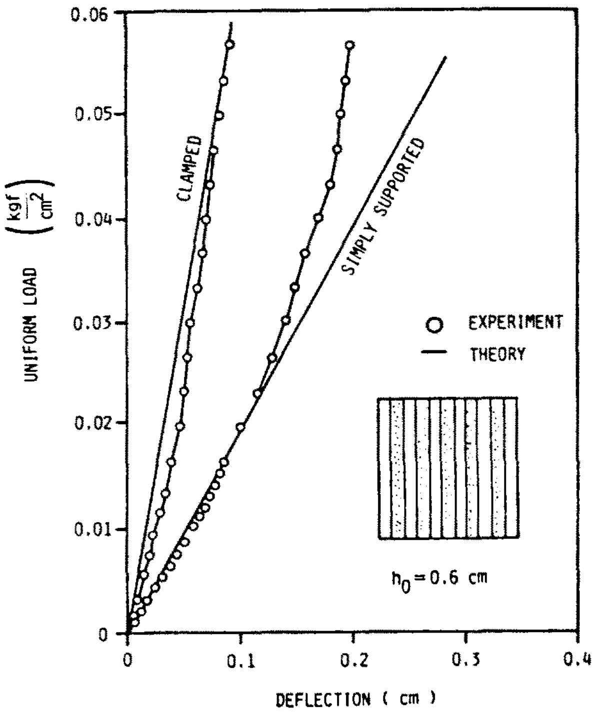


Fig. 12. Relationship between the lateral load and deflection for Type 2.

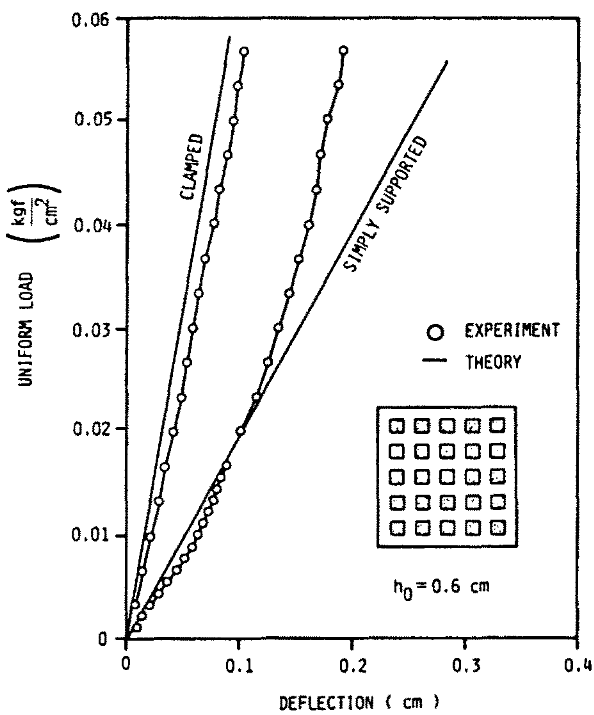


Fig. 13. Relationship between the lateral load and deflection for Type 3.

6. CONCLUSIONS

A general analytical method for isotropic rectangular plates with arbitrarily-positioned voids has been proposed by means of an extended Dirac function. The static solutions for simply-supported and clamped plates with voids were presented by means of the Galerkin method. The exactness of the proposed solutions was demonstrated by comparing the numerical results with the results of the finite element method, the results of equivalent plate analogy and the experimental results.

For the sake of simplicity, this paper disregards the transverse shear deformation and the local deformation of the top and bottom platelets of the void. When the cross-section or number of voids becomes large, it will be necessary to consider these deformations. The transverse shear deformation is considered by replacing the Kirchhoff–Love hypotheses with Mindlin's plate theory (Hinton and Owen, 1984). The local deformation of the top and bottom platelets of the void can be considered by using the frame theory. However, in practice, occurrence of the local deformation should be restricted.

Each void was assumed to be a rectangular parallelepiped for simplicity's sake, but it is relatively easy to extend the proposed theory to a void with circular or symmetric cross-section.

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